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Parametric (Anti-) Self-Dual Variables and a Related Parametric Yang–Mills-Like Action in Four-Dimensional Gravity

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Internal and external parametric dual transformations as well as (anti-) self-dual variables in four-dimensional gravity are presented in a unified way via a parameter. The double complex (anti-) self-dual variables including the double complex Ashtekar variables are studied by using the double complex function method. Concretely, a parametric Yang–Mills-like action is proposed and the analytic continuation from a double action to the Euclidean action is discussed. Some results of previous gravitational theories are extended to a parametric form.

1. INTRODUCTION AND PRELIMINARIES

The Ashtekar variables (Ashtekar, 1986) have been considered as fundamental variables in the description of classical and quantum gravity (Jacobson and Smolin, 1987) since these (anti-) self-dual variables lead to a much simpler Hamiltonian constraint than the Arnowitt–Deser–Misner (ADM) formulation (Arnowitt *et al.*, 1962). Moreover, the Ashtekar formulation has provided us with a new way to study gravity from a nonperturbative point of view. In spite of the success of the formulation, there are still several problems that the Ashtekar program has to face. The most important one is the issue of the reality conditions. As is well known, the reality conditions must be imposed on the complex (or elliptic complex) Ashtekar variables in order to recover the usual real formulation of general relativity for space-times with Lorentzian signature. Thus, some effort (Barbero, 1994, 1995) has been made to overcome the drawback of introducing complex variables. On the other hand, a number of authors (e.g., Barbero, 1994, 1995; Soo, 1995) have

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discussed those (anti-) self-dual variables corresponding to the Euclidean signature of the metric, which must be real in previous theories. But, in fact, they may be complex (i.e., hyperbolic complex) variables, as we shall see below. Thus the question arises whether there is a unified method to deal simultaneously with real and complex (anti-) self-dual variables for Lorentzian and Euclidean signatures in four-dimensional gravity. We will answer this question affirmatively by using a parameter which is used to control the space-time signature. Moreover, we can treat, in a unified way, ordinary complex and hyperbolic complex (anti-) self-dual variables by means of the double complex function method (DCFM). In fact, as discussed elsewhere (Zhong, 1985; Hucks, 1993; Wu, 1994), DCFM and the hyperbolic complex structure have been applied to many fields in physics. Hence it should be possible to apply DCFM to complex (anti-) self-dual gravitational theory (SDGT). In this paper we will show that real, ordinary complex and hyperbolic complex (anti-) self-dual variables in four-dimensional gravity can be expressed in a unified way via a parameter, and DCFM not only can treat the complex Ashtekar variables, which may provide us with a way to study simultaneously Lorentzian and Euclidean gravity from a nonperturbative point of view, but also give some new results, and extend some given results (Soo, 1995) into a parametric form.

In order to stress the role of the parameter β to be introduced and to apply DCFM to complex SDGT, here we introduce a parametric real 4-manifold $M(\beta) = (M_E, M(J))$ instead of the usual real 4-manifold. For any point $p \in M(\beta)$ there is an internal space which contains a real internal Euclidean space for the real 4-manifold M_E and a double complexified internal space, i.e., an internal Minkowskian space and an internal Euclidean space for the double real 4-manifold $M(J)$.

We begin with a brief introduction of DCFM. Let J denote the double imaginary unit, i.e., $J = i(i^2 = -1)$ or $J = \varepsilon(\varepsilon^2 = +1, \varepsilon \neq 1)$. $Z(J) = a(J) + b(J)$ is called a double complex number, where $a(J)$ and $b(J)$ are double real numbers. Sometimes $Z(J)$ may be directly written as $Z(J) = (Z_C, Z_H)$, where $Z_C = Z(J = i)$, $Z_H = (J = \varepsilon)$, which are called ordinary complex and hyperbolic complex numbers, respectively. On the other hand, we know that in the connection dynamics of gravity of four dimensions, the connection one-form can be decomposed into self-dual and anti-self-dual parts. The (anti-) self-dual part of the connection plays a key role in the Ashtekar formulation. Moreover, the connection one-form can be self-dual or anti-self-dual with respect to only its internal indices. But the decomposition into self-dual parts of the curvature two-form G can be dualized with respect to internal and external indices. Generally speaking, if a two-form carries a pair of antisymmetric internal indices AB, with each index taking values from 0 to 3, it is possible to consider the internal and external duality transformations $G*$ and $*G$. As has been pointed out (Soo, 1995), the squares of the internal and external dual operators acting on G are plus or minus the identities, i.e.,

$$G** = **G = \pm G \tag{1}$$

where plus and minus signs respectively correspond to the Euclidean and the Lorentzian signature, $\eta_E = diag(+1, +1, +1, +1)$ and $\eta_L = diag(-1, +1, +1, +1)$. It follows that the eigenvalues of the internal and external dual operators with Lorentzian signature are $\pm i$, and the eigenvalues with Euclidean signature are ± 1 and $\pm \varepsilon$. From this we can easily find that for the description of Euclidean gravity we can not only use real (anti-) self-dual variables which are familiar to us, but also use hyperbolic complex (anti-) self-dual variables.

2. INTERNAL AND EXTERNAL PARAMETRIC DUAL TRANSFORMATIONS AND (ANTI-) SELF-DUAL OPERATORS

According to the above discussion, we introduce a parameter β to denote the different eigenvalues of the dual operator. Of course, here β can only take as values the real unit 1 and the double imaginary unit J. This means the dual transformation corresponding to its different eigenvalues β is parametric, i.e., $* = *(\beta)$. Hence, the definition of the internal and external parametric dual transformations can be respectively given as follows:

$$G*_{AB\mu\nu}(\beta) := \tfrac{1}{2}\varepsilon_{AB}^{KL}(\beta)G_{KL\mu\nu}(\beta) \tag{2}$$

$$*G_{AB\mu\nu}(\beta) := \tfrac{1}{2}e(\beta)\varepsilon_{\mu\nu}^{\lambda\sigma}(\beta)G_{AB\lambda\sigma}(\beta) \tag{3}$$

When taking $\beta = 1$, the star $*$ is the usual dual transformation for the Euclidean signature, which here denote by $*_E$. But if $\beta = J$, the star $*$ is doubled, i.e., $* = *(J) = (*_C, *_H)$, which respectively correspond to the Lorentzian and the Euclidean signature. The explicit forms are

$$G*_{AB\mu\nu}(J) := \tfrac{1}{2}\varepsilon_{AB}^{KL}(J)G_{KL\mu\nu}(J) \tag{4}$$

$$*G_{AB\mu\nu}(J) := \tfrac{1}{2}e(J)\varepsilon_{\mu\nu}^{\lambda\sigma}(J)G_{AB\lambda\sigma}(J) \tag{5}$$

The internal indices of $\varepsilon_{AB}^{KL}(J)$ and the external indices of $\varepsilon_{\mu\nu}^{\lambda\sigma}(J)$ are raised by $\eta_{AB}(J) = (\eta_{AB}(_C), \eta_{AB}(_H)) = (\eta_E, \eta_L)$ and $g_{\mu\nu}(J)$, and $e(J)$ is the determinant of the vierbein, $e^A(J) = e_\mu^A(J)dx^\mu$. If $G_{AB\mu\nu}(\beta)$ satisfies

$$G*(\beta) = \pm\beta G(\beta) \qquad \text{and} \qquad *G(\beta) = \pm\beta G(\beta) \tag{6}$$

then $G(\beta)$ is called the internal and external parametric self-dual or anti-self-dual, respectively. Notice that when $\beta = 1$, G satisfying $G* = \pm G$ ($*G = \pm G$) is a real internal (external) self-dual or anti-self-dual variable for the Euclidean signature, and if $\beta = J$, $G(J)$ satisfying $G*(J) = \pm JG(J)$ [$*G(J) = \pm JG(J)$] is called a double complex internal (external) self-dual or anti-self-dual variable, e.g., $G(J = i) = G_C$ is called an ordinary complex one corresponding to Lorentzian signature, and $G(J = c) = G_H$ is here called a hyperbolic complex one corresponding to Euclidean signature.

According to Chee (1996), a parametric two-form $G(\beta)$ can be decomposed into four parts:

$$G(\beta) = {}^{+}G^{+}(\beta) + {}^{-}G^{+}(\beta) + {}^{+}G^{-}(\beta) + {}^{-}G^{-}(\beta) \tag{7}$$

where ${}^{+}G^{+}(\beta)$ [${}^{-}G^{-}(\beta)$] is self-dual (anti-self-dual) with respect to both internal and external indices, and ${}^{+}G^{-}(\beta)$ [${}^{-}G^{+}(\beta)$] external self-dual (anti-self-dual) and internal anti-self-dual (self-dual). The parametric internal and external self-dual operators + (anti-self-dual operators −) are defined to be

$$G^{\pm}(\beta) := \tfrac{1}{2}(G*(\beta) \pm \beta G(\beta)) \tag{8}$$

$$^{\pm}G(\beta) := \tfrac{1}{2}(*G(\beta) \pm \beta G(\beta)) \tag{9}$$

It follows that $G^{\pm}(\beta = 1) = G_E^{\pm}$ is a real self-dual (or anti-self-dual) variable for the Euclidean signature, and $G^{\pm}(\beta = i) = G_C^{\pm}$ and $G^{\pm}(\beta = \varepsilon) = G_H^{\pm}$ are ordinary and hyperbolic complex self-dual (or anti-self-dual) variables for the Lorentzian and the Euclidean signature, respectively. In addition, each of the four parts in expression (7) can be expressed by $G(\beta)$. For example,

$$\begin{aligned} {}^{-}G^{-}(\beta) &= \tfrac{1}{2}(*G^{-}(\beta) - \beta G^{-}(\beta)) \\ &= \tfrac{1}{4}(*G*(\beta) + \beta^2 G(\beta) - \beta G(\beta) - \beta G*(\beta)) \end{aligned} \tag{10}$$

It can be verified that the definitions given above are true for arbitrary an n-form of a parametric internal space at a point on a $2n$-dimensional real manifold.

3. THE RELATED PARAMETRIC YANG–MILLS-LIKE ACTION

It is a matter of convention to use either self-dual or anti-self-dual variables. So we choose to use anti-self-dual variables for all our discussions and adopt the convention that upper case Latin indices denoting internal indices run from 0 to 3, while lower case indices run from 1 to 3.

Let $\Gamma_{AB}(\beta) = -\Gamma_{BA}(\beta)$ be a parametric connection one-form, and

$\Gamma^{-}_{AB}(\beta) = \frac{1}{2}(\Gamma*_{AB}(\beta) - \beta\Gamma_{AB}(\beta))$. It can be verified that the curvature two-form of $\Gamma^{-}_{AB}(\beta)$,

$$F^{-}_{AB}(\beta) = d\Gamma^{-}_{AB}(\beta) + \Gamma^{-}_{AD}(\beta) \wedge \Gamma_{B}^{-D}(\beta) \tag{11}$$

satisfies

$$F^{-}_{AB}(\beta) = \tfrac{1}{2}(F*_{AB}(\beta) - \beta F_{AB}(\beta)) \tag{12}$$

where

$$F_{AB}(\beta) = d\Gamma_{AB}(\beta) + \Gamma_{AD}(\beta) \wedge \Gamma^{D}_{B}(\beta) \tag{13}$$

In addition, here we introduce a parametric two-form $\Sigma_{AB}(\beta) = e_A(\beta) \wedge e_B(\beta)$. For it, internal and external dual transformations are the same, i.e.,

$$\Sigma*_{AB}(\beta) = *\Sigma_{AB}(\beta) \tag{14}$$

Now we construct the proposed parametric Yang–Mills-like action as follows:

$$\begin{aligned} S(\beta) &= S(e(\beta), \Gamma^{-}(\beta)) \\ &= \beta^2 \int_{M(\beta)} \left[\frac{\beta}{g}\,{}^{-}F^{-}_{AB}(\beta) + \beta^2 \frac{g}{16\pi G}\Sigma^{-}_{AB}(\beta)\right] \\ &\quad \wedge * \left[\frac{\beta}{g}\,{}^{-}F^{-AB}(\beta) + \beta^2 \frac{g}{16\pi G}\Sigma^{-AB}(\beta)\right] \end{aligned} \tag{15}$$

where ${}^{-}F^{-}_{AB}(\beta) = \frac{1}{2}(*F^{-}_{AB}(\beta) - \beta F^{-}_{AB}(\beta))$, g and G are constants here. If we let

$$\Omega(e(\beta), \Gamma^{-}(\beta)) = \frac{\beta -}{g} F^{-}_{AB}(\beta) + \beta^2 \frac{g}{16\pi G}\Sigma^{-}_{AB}(\beta) \tag{16}$$

we can rewrite equation (15) as

$$\begin{aligned} S(\beta) = S(e(\beta), \Gamma^{-}(\beta)) &= \beta^2 \int_{M(\beta)} Tr(\Omega \wedge *\Omega) = -\beta^3 \int_{M(\beta)} \mathrm{Tr}(\Omega \wedge \Omega) \\ &= -\beta^3 \int_{M(\beta)} \left\{ \begin{array}{l} \frac{1}{2g^2}[F^{-}_{AB}(\beta) \wedge F^{-AB}(\beta) - \beta^3 *(\beta) F^{-}_{AB}(\beta) \wedge F^{-AB}(\beta)] \\ -\frac{1}{8\pi G} F^{-}_{AB}(\beta) \wedge \Sigma^{-AB}(\beta) + \frac{g^2}{(16\pi G)^2}\Sigma^{-}_{AB}(\beta) \wedge \Sigma^{-AB}(\beta) \end{array} \right\} \end{aligned} \tag{17}$$

From equation (17) we can see the following.

1. Since Ω is anti-self-dual with respect to both internal and external indices, equation (17) adds a parametric Yang–Mills action $\int_{M(\beta)} \mathrm{Tr}(F(\beta) \wedge *F^{-}(\beta))$, in contrast with the proposed action in the gauge theory of the de Sitter group (Nieto *et al.*, 1994). Further, we can find that (17) gives similar results with the internal and external anti-self-dual parts of $\mathrm{Tr}(*\Omega \wedge \Omega)$ and $\mathrm{Tr}(\Omega \wedge \Omega *)$ when the torsion vanishes (Chee, 1996).

2. In the two cases of $\beta = 1$ and $\beta = i$ in equation (17) we can respectively get the positive-semidefinite action $S_E = S(\beta = 1)$ and the corresponding Lorentzian action $S_L = S(\beta = i)$ given by Soo (1995). But when taking $\beta = \varepsilon$, we find $S_H = S(\beta = \varepsilon)$ is a new action, which is

$$S_H = -\varepsilon \int_{M_H} \left\{ \begin{array}{l} \frac{1}{2g^2}[F^{-}_{AB}(H) \wedge F^{-AB}(H) - \varepsilon *(H) F^{-}_{AB}(H) \wedge F^{-AB}(H)] \\ -\frac{1}{8\pi G} F^{-}_{AB}(H) \wedge \Sigma^{-AB}(H) + \frac{g^2}{(16\pi G)^2} \Sigma^{-}_{AB}(H) \wedge \Sigma^{-AB}(H) \end{array} \right\} \tag{18}$$

Here it is called the hyperbolic Euclidean action. Obviously it differs from the Euclidean action S_E. It should be noted that the third term in (18) corresponds to the Ashtekar action (Soo, 1995), so we call it the hyperbolic Ashtekar action. It follows that $S(J)$ contains a double Ashtekar action, i.e., the ordinary and hyperbolic Ashtekar actions. However, whether we can obtain the equations of motion of the double action as well as put the double complex SDGT in Hamiltonian form is the subject of forthcoming papers.

4. THE ANALYTIC CONTINUATION FROM THE DOUBLE ACTION TO THE EUCLIDEAN ACTION

In the following, we shall show that it is possible to continue the double action $S(J) = (S_c, S_H)$ to the Euclidean action S_E by a double Wick rotation (DWR), and the double analytic continuation (DAC) from $S(J)$ to S_E has the property that $\exp(J^3 S(J)) = \exp(S_E)$. Moreover, the self-duality of the fields $\Gamma^{-}(J)$, $F^{-}(J)$, and $\Sigma^{-}(J)$ with respect to the double internal $*(J)$ as well as the self-duality of ${}^{-}F^{-}$ and $\Sigma^{-}(J)$ with respect to the double external $*(J)$ can be preserved in the course of the continuation.

A double Wick rotation DWR which includes both the ordinary and hyperbolic Wick rotation, i.e., DWR = (CWR, HWR) with $e_0(J) = J^3(e_E)_0$ and $e_a(J) = (e_E)_a$ will result in the metric having Euclidean signature $\eta_E = \mathrm{diag}(+1, +1, +1, +1)$. The corresponding changes induced here are

$$J^2\Sigma^{-0a}(J) = \Sigma^{-}_{0a}(J) \mapsto \Sigma^{-0a}_{E} = (\Sigma^{-}_{E})_{0a} \tag{19}$$

$$J^2 F^{-0}{}_{a}(J) = F^{-}_{0a}(J) \mapsto F^{-0}_{E}{}_{a} = (F^{-}_{E})_{0a} \tag{20}$$

$$J^3 *(J) \mapsto *_E \tag{21}$$

To obtain the continuation explicitly, we note that $\Sigma^{-}_{bc}(J) = -J^3\varepsilon^{0a}_{bc}(J)\Sigma^{-}_{0a}(J)$ and $F^{-}_{bc}(J) = -J^3\varepsilon^{0}_{bc}(J)F^{-}_{0a}(J)$. Therefore the double action becomes

$$\begin{aligned} S(J) &= S(e(J),\ \Gamma^{-}_{0a}(J)) \\ &= -J^3\int_{M(J)}\left\{\begin{array}{l} \frac{8}{2g^2}[F^{-}_{0a}(J)\wedge F^{-0a}(J) - J^3 *(J)F^{-}_{0a}(J)\wedge F^{-0a}(J)] \\ -\frac{1}{8\pi G}8F^{-}_{0a}(J)\wedge\Sigma^{-0a}(J) + \frac{g^2}{(16\pi G)^2}\Sigma^{-}_{0a}(J)\wedge\Sigma^{-0a}(J)\end{array}\right\} \end{aligned} \tag{22}$$

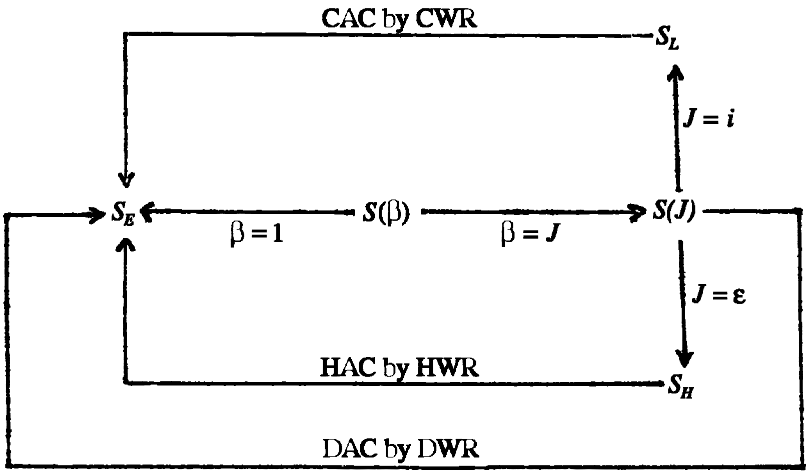
Thus

$$-J^3S(J) = \int_{M(J)}\left\{\begin{array}{l} \frac{8}{2g^2}[F^{-}_{0a}(J)\wedge J^2F^{-0a}(J) - J^3 *(J)F^{-}_{0a}(J)\wedge J^2F^{-0a}(J)] \\ -\frac{1}{8\pi G}8F_{0a}(J)\wedge J^2\Sigma^{-0a}(J) + \frac{g^2}{(16\pi G)^2}\Sigma_{0a}(J)\wedge J^2\Sigma^{-0a}(J)\end{array}\right\} \tag{23}$$

is continued to

$$\begin{aligned} &\int_{M(E)}\left\{\begin{array}{l} \frac{8}{2g^2}[F^{-}_{0a}(E)\wedge F^{-0a}(E) - *(E)F^{-}_{0a}(E)\wedge F^{-0a}(E)] \\ -\frac{1}{8\pi G}8F^{-}_{0a}(E)\wedge J^2\Sigma^{-0a}(E) + \frac{g^2}{(16\pi G)^2}\Sigma^{-}_{0a}(E)\wedge J^2\Sigma^{-0a}(E)\end{array}\right\} \\ &= \int_{M_E}\left\{\begin{array}{l} \frac{8}{2g^2}[(F^{-}_{E})_{0a}\wedge F^{-0a}_{E} - *_E(F^{-}_{E})_{0a}\wedge F^{-0a}_{E}] \\ -\frac{1}{8\pi G}8(F^{-}_{E})_{0a}\wedge\Sigma^{-0a}_{E} + \frac{g^2}{(16\pi G)^2}(\Sigma^{-}_{E})_{0a}\wedge\Sigma^{-0a}_{E}\end{array}\right\} \\ &= \int_{M_E}\left\{\begin{array}{l} \frac{1}{2g^2}[(F^{-}_{E})_{AB}\wedge F^{-AB}_{E} - *_E(F^{-}_{E})_{AB}\wedge E^{-AB}_{E}] \\ -\frac{1}{8\pi G}(F^{-}_{E})_{AB}\wedge\Sigma^{-AB}_{E} + \frac{g^2}{(16\pi G)^2}(\Sigma^{-}_{E})_{AB}\wedge\Sigma^{-Ab}_{E}\end{array}\right\} \\ &= S_E \end{aligned} \tag{24}$$

where S_E is precisely the Euclidean action. So we indeed have a double analytic continuation DAC=(CAC, HAC) from $\exp(J^3\, S(J))$ to $\exp(S_E)$. The above discussion about the DAC can be diagrammed as follows:



(25)

From (25) we easily see that the continuation from Lorentzian to Euclidean signature (Soo, 1995) is just the CAC by CWR in (25). Obviously, it is only half of our results.

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